New Multiplicative Structures on Frobenius Algebras

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Introduction: Frobenius algebras

- In the 1930s, Frobenius algebras were first prominently studied (for representation theory).
- Around 1940, Nakayama discovered important duality theories that helped characterize Frobenius algebras and contributed to:
  - Number theory,
  - Algebraic geometry,
  - Combinatorics,
  - Homological algebra.
- Recently, their correspondence to Topological Quantum Field Theories has revived their popularity.

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- TQFTs combine classical field theory, special relativity, and quantum mechanics.
- Aside from physics, TQFTs' applications include:
  - Topological invariants,
  - Knot theory,
  - Four-dimensional manifolds in algebraic topology,
  - Moduli spaces in algebraic geometry.

Our aims

- The tensor product gives a multiplicative structure on Frobenius algebras.
- Are there new, nontrivial multiplicative structures preserving commutativity?
- Our approach: twisting the tensor product.

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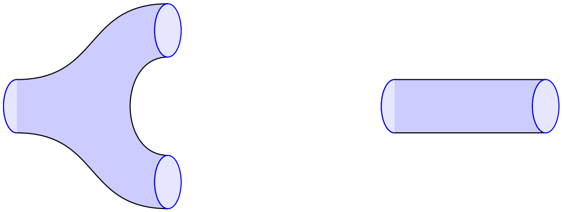
In other words, we essentially use that vector spaces form a *symmetric monoidal category.*

Cobordisms

Definition

A 2-**cobordism** is a closed, oriented 2-manifold linking the disjoint unions of some number of circles.

Example



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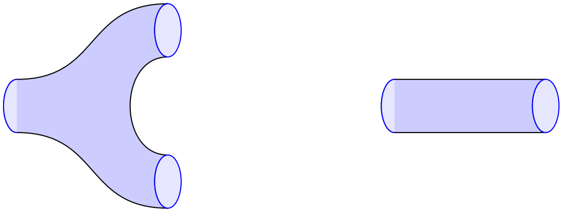
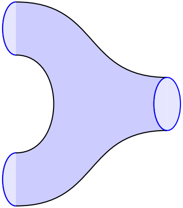


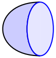
We associate cobordisms to k-linear maps between tensor powers of a vector space A.

Algebras

Definition

An **algebra** is a vector space A along with multiplication and unit as below,





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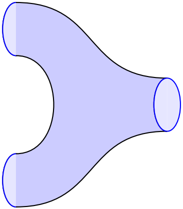
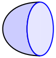
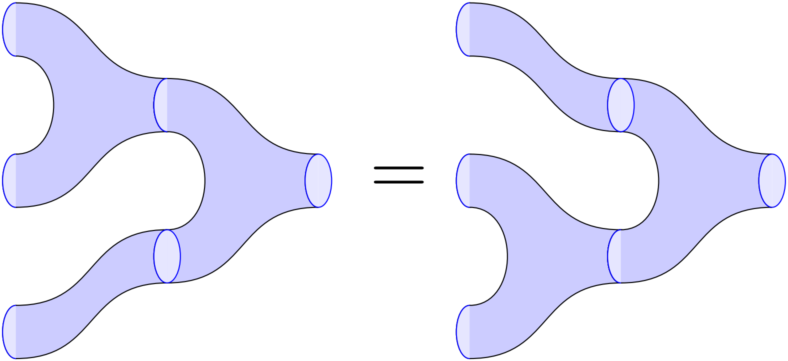


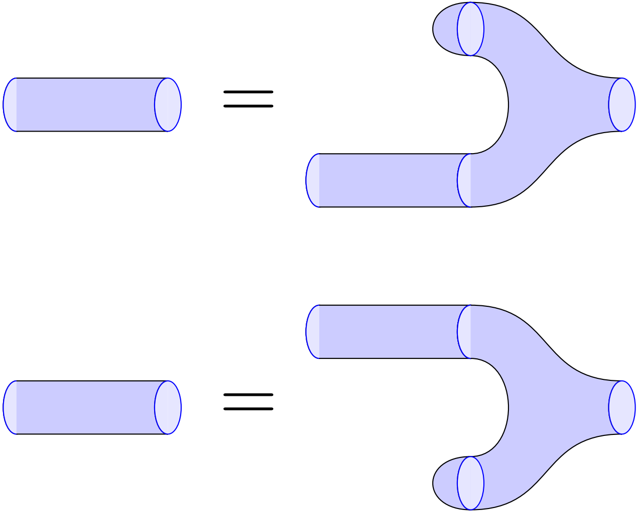


such that multiplication is associative and unital.

Algebras: assocativity



Algebras: unitality



Frobenius algebras

Definition

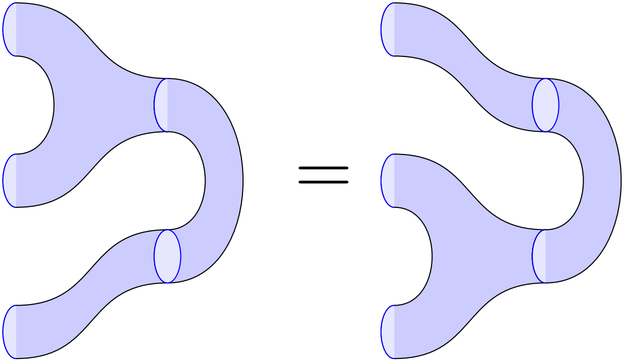
A **Frobenius algebra** is an algebra with a pairing as below,

which is associative and nondegenerate.

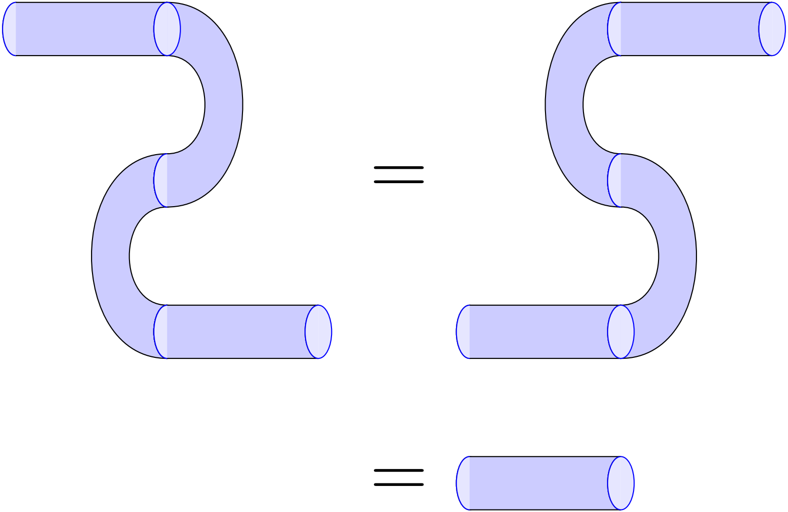
Remark

Nondegeneracy requires the existence of a copairing, drawn on the right above.

Frobenius algebras: associativity



Frobenius algebras: nondegeneracy



Examples of Frobenius algebras

These examples are in terms of a counit $\epsilon : A \to k$; from this, we can define a pairing that is multiplication follows by the counit.

- The algebra $A = k$ with counit ϵ given by multiplication,
- A finite field extension A over k with any k-linear counit $\epsilon : A \to k$,
- The matrix algebra $\mathrm{Mat}_n(k)$ with the trace map as the counit,
- For a group G, the group algebra kG of linear combinations of group elements with counit $\epsilon : g \mapsto \delta_{ge}$,

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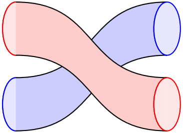
- For a group G, the group algebra kG of linear combinations of group elements with counit $\epsilon : g \mapsto \delta_{ge}$,
- The algebra $k[t]/(t^2 - 1)$ with counit $\epsilon : 1 \mapsto 1$, $t \mapsto 0$.
- The algebra $k[t]/t^2$ with counit $\epsilon : 1 \mapsto 0$, $t \mapsto 1$.

The trivial twisting map

Color A red, B blue, and represent σ by the below cobordism.



The standard tensor product

Definition

For Frobenius algebras A and B, the standard tensor product $A \otimes_\sigma B$ is $A \otimes B$ equipped with a unit, multiplication, and pairing as shown.

The standard tensor product: unit

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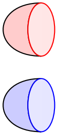

The standard tensor product: multiplication

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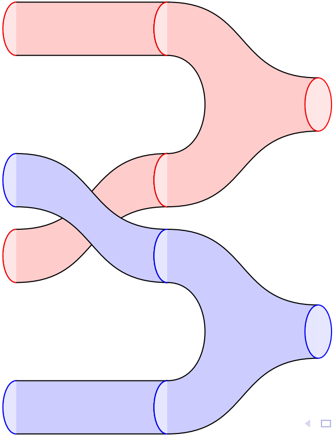

The standard tensor product: pairing

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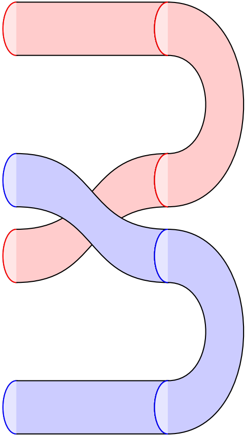

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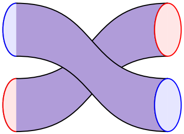
Proposition

For all Frobenius algebras A and B, $A \otimes_\sigma B$ is a Frobenius algebra.

Warped tensor products of Frobenius algebras

Definition

Let $\gamma : B \otimes A \to A \otimes B$ be a warp as shown on the left. Define $A \otimes_\gamma B$ as $A \otimes_\sigma B$ with the pairing changed, as shown below on the right.



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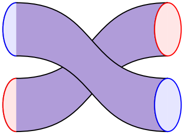
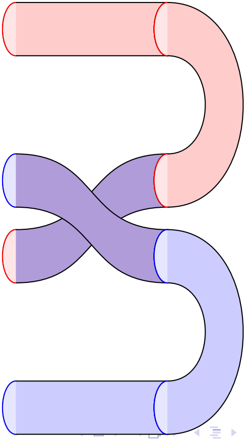
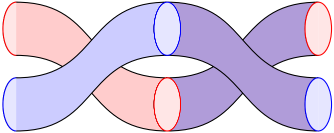




Result

Theorem

Let A and B be Frobenius algebras, and let $\gamma : B \otimes A \to A \otimes B$ be a warp.
Then $A \otimes_\gamma B$ is a Frobenius algebra with the natural copairing if and only if $\gamma \circ \sigma$ is multiplication by a central, invertible element of $A \otimes B$.



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Thank you!